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L_∞-Upper Bound of L_2-Projections onto Splines at a Geometric Mesh

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For an integer $k \geqslant 1$ and a geometric mesh $(q^i)_{-\infty}^{\infty}$ with $q \in (0, \infty)$, let

$$M_{i,k}(x) := k[q^i,..., q^{i+k}](\cdot - x)_+^{k-1},$$
$$N_{i,k}(x) := (q^{i+k} - q^i)\, M_{i,k}(x)/k,$$

and let $A_k(q)$ be the Gram matrix $(\int M_{i,k} N_{j,k})_{i,j \in \mathbb{Z}}$. It is known that $\|A_k(q)^{-1}\|_\infty$ is bounded independently of q. In this paper it is shown that $\|A_k(q)^{-1}\|_\infty$ is strictly decreasing for q in $[1, \infty)$. In particular, the sharp upper bound and lower bound for $A_k(q)^{-1}$ are obtained:

$$2k - 1 \leqslant \|A_k(q)^{-1}\|_\infty \leqslant \left(\frac{\pi}{2}\right)^{2k} \left\{\sum_{j \in \mathbb{Z}} (1 + 2j)^{-2k}\right\}^{-1}$$

for all $q \in (0, \infty)$.

1. INTRODUCTION

Let $\mathbf{x} := (x_i)_{-\infty}^{\infty}$ be a strictly increasing biinfinite sequence with $x_{\pm\infty} := \lim_{i \to \pm\infty} x_i$ and $I := (x_{-\infty}, x_{+\infty})$. Further, let

$$S := m\mathbb{S}_{k,\mathbf{x}}(I) := \{f \in C^{k-2}(I) \cap L_\infty(I);\ f|_{(x_i, x_{i+1})} \text{ is a polynomial of degree} < k\}$$

be the normed linear space of bounded polynomial splines of order k with breakpoint sequence $\mathbf{x}$ and norm $\|f\| := \sup_{x \in I} |f(x)|$. We shall be concerned with P_S, the orthogonal projector onto S with respect to the ordinary inner product

$$(f, g) := \int_I f(x)\, g(x)\, dx,$$

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but restricted to $L_\infty(I)$. We want to bound its norm

$$\|P_S\|_\infty := \sup_{f \in L_\infty(I)} \|P_S f\|/\|f\|_\infty.$$

In 1973, de Boor raised the following

Conjecture [1].

$$\sup_{\mathbf{x}} \|P_S\|_\infty \leqslant \text{const}_k < \infty.$$

This conjecture has been verified for $k = 1, 2, 3, 4$ (see de Boor [3] and the references cited there). De Boor [2] also obtained a bound of P_S in terms of a global mesh ratio. In general, however, this conjecture seems hard to solve. For a geometric mesh $\mathbf{x}$, Höllig [8] recently proved the boundedness of P_S. Later on, Feng and Kozak [6] reproved this result. Before recalling some results of theirs, we need to introduce some notations. For the mesh $\mathbf{x} = (x_i)_{-\infty}^{\infty}$, let

$$\begin{aligned} M_{i,k}(x) &:= k[x_i, ..., x_{i+k}](\cdot - x)_+^{k-1} \\ N_{i,k}(x) &:= ([x_{i+1}, ..., x_{i+k}] - [x_i, ..., x_{i+k-1}])(\cdot - x)_+^{k-1} \\ &= (x_{i+k} - x_i)\, M_{i,k}(x)/k. \end{aligned}$$

Set

$$A_k(i, j) := \int M_{i,k} N_{j,k} \qquad \text{for} \quad i, j \in \mathbb{Z}.$$

Let $A_k \in \mathbb{R}^{\mathbb{Z} \times \mathbb{Z}}$ be the biinfinite matrix given by the rule

$$(i, j) \to A_k(i, j) \qquad \text{for} \quad (i, j) \in \mathbb{Z} \times \mathbb{Z}.$$

It was shown by de Boor [1] that

$$D_k^{-2} \|A_k^{-1}\|_\infty \leqslant \|P_S\|_\infty \leqslant \|A_k^{-1}\|_\infty,$$

where D_k is a constant depending only on k. Thus bounding P_S is equivalent to bounding A_k^{-1}.

Let us restrict ourselves now to a particular case where $\mathbf{x}$ is a geometric mesh: $\mathbf{x} := (q^i)_{-\infty}^{\infty}$ for some $q \in [1, \infty)$ (note that the case $q \in (0, 1]$ is symmetric to the case $q \in [1, \infty)$; see [6, 8]). Spline interpolation at a geometric mesh was first investigated by Micchelli [9], who based his argument on the properties of the so-called generalized Euler–Frobenius polynomials. Later on, Feng and Kozak [6] developed such a consideration. Earlier, and in a different way, Höllig [8] made a more precise investigation into the boundedness of L_2-projections onto splines on a geometric mesh. In particular, he got the following elegant result (see [8, Theorem 5]):

THEOREM A. *For a geometric mesh* $\mathbf{x} := (q^i)_{-\infty}^{\infty}$ *with* $q \in (0, \infty)$, *let* $A_k(q)$ *be the biinfinite matrix* $(\int M_{i,k} N_{j,k})_{i,j \in Z}$. *Then*

$$\|A_k(q)^{-1}\|_\infty = |\Omega_k(q)|^{-1}, \tag{1}$$

where

$$\Omega_k(q) := 2k!\,(k-1)!\,t^{2k-1} \prod_{\nu=1}^{k} \frac{q^\nu + 1}{q^\nu - 1} \prod_{\nu=1}^{k-1} \frac{q^\nu + 1}{q^\nu - 1} \times \sum_{j \in Z} \prod_{\nu=1}^{k} \frac{1}{[\pi(1+2j)]^2 + (\nu t)^2} \tag{2}$$

with $t := \log q$. *Moreover,*

$$\lim_{q \to 1} \Omega_k(q) = \left(\frac{2}{\pi}\right)^{2k} \left\{\sum_{j \in Z} (1+2j)^{-2k}\right\} \tag{3}$$

$$\lim_{q \to \infty} \Omega_k(q) = \frac{1}{2k-1}. \tag{4}$$

Based on numerical evidence, de Boor raised the following

Conjecture. $\Omega_k(q)$ *is a monotone increasing function on* $[1, \infty)$.

This conjecture was verified for $k \leqslant q$ by Feng and Kozak [7]. They also showed that $\Omega_k(q) \leqslant 1/(2k-1)$ in the same paper.

The purpose of this paper is to confirm the above conjecture. Thus we have

THEOREM 1. $\Omega_k(q)$ *is a monotone increasing function on* $[1, \infty)$. *In particular,*

$$2k - 1 \leqslant \|A_k(q)^{-1}\|_\infty \leqslant \left(\frac{\pi}{2}\right)^{2k} \left\{\sum_{j \in Z} (1+2j)^{-2k}\right\}^{-1}. \tag{5}$$

Note that $\Omega_1(q) \equiv 1$ and that $\Omega_2(q) \equiv \frac{1}{3}$ in terms of a straightforward calculation. Hence we can restrict ourselves to the case $k \geqslant 3$ from now on.

In Section 2, we shall give an alternative proof of Theorem A. Sections 3 and 4 will be devoted to proving the monotonicity of $\Omega_k(q)$ for $q \in [1, 20]$ and $q \in [20, \infty)$, respectively.

2. The Bound for $A_k(q)^{-1}$

As before, $\mathbf{x} = (q^i)_{-\infty}^{\infty}$ is a geometric mesh with $q \in (1, \infty)$ and $t = \log q$. Consider

$$\phi_0(x) := [0, 1, \ldots, (2k-1)]_z\, x^z/(q^z + q^k) \qquad \text{for} \quad x \in [1, q].$$

It is easy to verify that

$$\begin{aligned} q^l \phi_0^{(l)}(q) + q^k \phi_0^{(l)}(1) &= [0, 1, \ldots, (2k-1)]_z \,\{z(z-1) \cdots (z-l+1)\} \\ &= 0, \qquad \text{for} \quad l = 1, \ldots, 2k-2, \\ &= 1, \qquad \text{for} \quad l = 2k-1. \end{aligned} \tag{6}$$

Since ϕ_0 is a polynomial of degree $2k-1$, $\phi_0^{(2k-1)}$ is constant in $[1, q]$. Hence (6) yields that

$$\phi_0^{(2k-1)}(x) = 1/(q^k + q^{2k-1}) \qquad \text{for} \quad x \in [1, q]. \tag{7}$$

Now we extend the domain of ϕ_0 to $(0, \infty)$ as

$$\phi(x) := (-q^k)^m \,\phi_0(q^{-m}x) \qquad \text{for} \quad q^m \leqslant x \leqslant q^{m+1}, \quad m \in \mathbb{Z}.$$

From (6) we assert that $\phi \in \mathbb{S}_{2k,\mathbf{x}}$, and that

$$\phi(q^m) = (-q^k)^m \,\phi_0(1), \qquad m \in \mathbb{Z}. \tag{8}$$

It follows that

$$\begin{aligned} [x_0, x_1, \ldots, x_{m-1}, x_m]\,\phi &= \frac{[x_1, \ldots, x_m]\,\phi - [x_0, \ldots, x_{m-1}]\,\phi}{x_m - x_0} \\ &= \frac{-q^{k-m+1}[x_0, \ldots, x_{m-1}]\,\phi - [x_0, \ldots, x_{m-1}]\,\phi}{q^m - 1} \\ &= -\frac{q^{k-m+1} + 1}{q^m - 1}\,[x_0, \ldots, x_{m-1}]\,\phi. \end{aligned}$$

By induction on m, we can obtain

$$\begin{aligned} [x_0, x_1, \ldots, x_k]\,\phi &= (-1)^k \left(\prod_{m=1}^{k} \frac{q^{k-m+1} + 1}{q^m - 1} \right) \phi_0(1) \\ &= (-1)^k \left(\prod_{m=1}^{k} \frac{q^m + 1}{q^m - 1} \right) \phi_0(1). \end{aligned} \tag{9}$$

From (8) we deduce that

$$[x_i,..., x_{i+k}]\,\phi = (-1)^i\,[x_0,..., x_k]\,\phi. \tag{10}$$

By Peano's theorem (see [4])

$$[x_i,..., x_{i+k}]\,\phi = \int M_{i,k}(x)\,\phi^{(k)}(x)/k!\,dx.$$

Now we get

$$\int M_{i,k}(x)\,\phi^{(k)}(x)/k!\,dx = (-1)^i\,(-1)^k \left(\prod_{m=1}^{k} \frac{q^m+1}{q^m-1}\right) \phi_0(1). \tag{11}$$

Obviously, $\phi^{(k)}/k! \in \mathbb{S}_{k,\mathbf{x}}$; hence $\phi^{(k)}/k!$ may be expanded in a B-spline series

$$\phi^{(k)}/k! = \sum \alpha_j N_{j,k};$$

however, $\phi^{(k)}(qx) = -\phi^{(k)}(x)$. Thus

$$\sum \alpha_j N_{j,k}(x) = -\sum \alpha_j N_{j,k}(qx) = -\sum \alpha_j N_{j-1,k}(x) = -\sum \alpha_{j+1} N_{j,k}(x).$$

By the uniqueness of B-spline expansion we assert that

$$\alpha_{j+1} = -\alpha_j, \qquad j \in \mathbb{Z}.$$

Thus we can write

$$\phi^{(k)}/k! = C \sum (-1)^j\,N_{j,k}, \tag{12}$$

where C is a constant to be determined. Now (11) and (12) together give

$$\sum_{j\in\mathbb{Z}} (-1)^j \int M_{i,k}(x)\,N_{j,k}(x)\,dx = (-1)^i\,C^{-1}(-1)^k \left(\prod_{m=1}^{k} \frac{q^m+1}{q^m-1}\right) \phi_0(1). \tag{13}$$

Let

$$\Omega_k(q) := C^{-1}(-1)^k \left(\prod_{m=1}^{k} \frac{q^m+1}{q^m-1}\right) \phi_0(1). \tag{14}$$

Then (see de Boor *et al.* [5])

$$\|A_k(q)^{-1}\|_\infty = |\Omega_k(q)|^{-1}.$$

It remains to determine C. Differentiate (12) $k-1$ times,

$$\phi^{(2k-1)}/k! = C\left(\sum (-1)^j\,N_{j,k}\right)^{(k-1)}.$$

One the one hand,

$$\phi^{(2k-1)}(x)/k! = \frac{1}{k!}\frac{1}{q^k + q^{2k-1}} \qquad \text{for} \quad x \in (1, q).$$

On the other hand (see [4]),

$$\begin{aligned}\left(\sum (-1)^j N_{j,k}\right)^{(k-1)} &= \frac{2(k-1)}{q^{k-1}-1}\frac{(q+1)(k-2)}{q^{k-2}-1}\cdots\frac{q^{k-2}+1}{q-1}\sum_{j\in\mathbb{Z}}(-1)^j (q^{k-1})^{-j} N_{j,1} \\ &= 2(k-1)!\frac{1}{q^{k-1}+1}\prod_{m=1}^{k-1}\frac{q^m+1}{q^m-1}\sum_{j\in\mathbb{Z}}(-1)^j (q^{k-1})^{-j} N_{j,1}.\end{aligned}$$

Thus, for $x \in (1, q)$

$$\left(\sum (-1)^j N_{j,k}\right)^{(k-1)}(x) = 2(k-1)!\frac{1}{q^{k-1}+1}\prod_{m=1}^{k-1}\frac{q^m+1}{q^m-1}.$$

From the above calculation we get

$$\begin{aligned}C^{-1} &= k!\,(q^k + q^{2k-1})\,2(k-1)!\frac{1}{q^{k-1}+1}\prod_{m=1}^{k-1}\frac{q^m+1}{q^m-1} \\ &= 2k!\,(k-1)!\,q^k\prod_{m=1}^{k-1}\frac{q^m+1}{q^m-1}.\end{aligned} \tag{15}$$

Finally, (14) and (15) yield that

$$\begin{aligned}\Omega_k(q) &= (-1)^k\,2k!\,(k-1)!\prod_{m=1}^{k}\frac{q^m+1}{q^m-1}\prod_{m=1}^{k-1}\frac{q^m+1}{q^m-1}\,q^k\phi_0(1) \\ &= (-1)^k\,2k!\,(k-1)!\prod_{m=1}^{k}\frac{q^m+1}{q^m-1}\prod_{m=1}^{k-1}\frac{q^m+1}{q^m-1} \\ &\quad\times q^k[0, 1,..., 2k-1]\frac{1}{q^{\cdot}+q^k}.\end{aligned} \tag{16}$$

We follow the procedure in [9] and use a well-known formula for the divided difference to get

$$\begin{aligned}&(-1)^k q^k[0, 1,..., 2k-1]\frac{1}{q^{\cdot}+q^k} \\ &= \frac{(-1)^k q^k}{2\pi i}\left[\int_{C_R} - \sum_{j=0}^{2k-1}\int_{C_{r_j}}\left(\prod_{m=0}^{2k-1}(z-m)(e^{z\log q}+q^k)\right)^{-1} dz,\right.\end{aligned} \tag{17}$$

where C_R and C_{r_j} stand for positively oriented circles with centers at 0 and j and radius R and r_j, where R is sufficiently large and r_j sufficiently small, $j = 0, 1, \ldots, 2k-1$.

Making $R \to \infty$, $r_j \to 0$ $(j = 0, 1, \ldots, 2k-1)$ in (17) and using the residue theorem we get

$$\begin{aligned}
&(-1)^k q^k [0, 1, \ldots, 2k-1] \frac{1}{q^{\cdot} + q^k} \\
&\quad = (-1)^k q^k (-1) \sum_{j \in \mathbb{Z}} \operatorname{Res}_{z = i(\pi + 2\pi j)/t + k} \left(\prod_{\nu=0}^{2k-1} (z - \nu)(e^{z \log g} + q^k) \right)^{-1} \\
&\quad = (-1)^k \sum_{j \in \mathbb{Z}} \left(t \prod_{\nu=0}^{2k-1} \left[\frac{i(\pi + 2\pi j)}{t} - \nu + k \right] \right)^{-1} \\
&\quad = \sum_{j \in \mathbb{Z}} t^{2k-1} \Big/ \left(\prod_{\nu=0}^{2k-1} [\pi + 2\pi j - i(k - \nu)\, t] \right) \\
&\quad = t^{2k-1} \sum_{j \in \mathbb{Z}} \prod_{\nu=1}^{k} \frac{1}{(\pi + 2\pi j)^2 + (\nu t)^2}.
\end{aligned}$$

Thus (2) is proved by substituting the above equality into (16). Then it is straightforward to verify (3). As to (4), we have

$$\begin{aligned}
\lim_{q \to \infty} \Omega_k(q) &= (-1)^k\, 2k!(k-1)!/(2k-1)! \lim_{q \to \infty} \sum_{l=0}^{2k-1} (-1)^{l+1} \binom{2k-1}{l} \frac{q^k}{q^l + q^k} \\
&= 2(-1)^k \Big/ \binom{2k-1}{k} \left[\sum_{l=0}^{k-1} (-1)^{l+1} \binom{2k-1}{k} \right. \\
&\qquad \left. + (-1)^{k+1} \frac{1}{2} \binom{2k-1}{k} \right] \\
&= 2(-1)^k \Big/ \binom{2k-1}{k} \left\{ \sum_{l=0}^{k-1} (-1)^{l+1} \left[\binom{2k-2}{l-1} + \binom{2k-2}{l} \right] \right. \\
&\qquad \left. + (-1)^{k+1} \frac{1}{2} \binom{2k-1}{k} \right\} \\
&= 2(-1)^k \Big/ \binom{2k-1}{k} \left[(-1)^k \binom{2k-2}{k-1} + (-1)^{k+1} \frac{1}{2} \binom{2k-1}{k} \right] \\
&= \frac{2k}{2k-1} - 1 = \frac{1}{2k-1}.
\end{aligned}$$

This ends the proof of Theorem A.

3. The Monotonicity of $\Omega_k(q)$ for $q \in [1, 20]$

Recall $t = \log q$. Let

$$f_k(t) := t^{2k-1} \prod_{\nu=1}^{k} \frac{e^{\nu t}+1}{e^{\nu t}-1} \prod_{\nu=1}^{k-1} \frac{e^{\nu t}+1}{e^{\nu t}-1} \sum_{j\in\mathbb{Z}} \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi j)^2+(\nu t)^2}.$$

Then $\Omega_k(e^t) = 2k!\,(k-1)!\,f_k(t)$. Consider $f_k'(t)/f_k(t)$. We have

$$\frac{f_k'(t)}{f_k(t)} = \sum_{j=0}^{\infty} \frac{1}{t}\, u_{k,j}(t) f_{k,j}(t), \tag{18}$$

where

$$u_{k,j}(t) := \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi j)^2+(\nu t)^2} \Big/ \left\{ \sum_{l=0}^{\infty} \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi l)^2+(\nu t)^2} \right\}$$

$$f_{k,j}(t) := 2k-1 + \left(\sum_{\nu=1}^{k-1} + \sum_{\nu=1}^{k} \right) \left(\frac{\nu t e^{\nu t}}{e^{\nu t}+1} - \frac{\nu t e^{\nu t}}{e^{\nu t}-1} \right)$$
$$- \sum_{\nu=1}^{k} \frac{2(\nu t)^2}{(\pi+2\pi j)^2+(\nu t)^2}. \tag{19}$$

If we can show that $f_k'(t)/f_k(t) \geqslant 0$ for $t \in [0, 3]$, then $\Omega_k'(q) \geqslant 0$ for $q \in [1, 20]$, because $e^3 > 20$. For this it suffices to show $f_{k,0}(t) \geqslant 0$, since $f_{k,j}(t) \geqslant f_{k,0}(t)$ ($j = 1, 2,\ldots$) from (19). Let us first make the following observation:

Proposition 1.

$$\frac{\pi^2}{\pi^2+(cx)^2} \geqslant \frac{2xe^x}{e^{2x}-1} \qquad \textit{for} \quad x \in [0, \infty) \quad \textit{and} \quad c \in [1, 5/4].$$

Proof. Each of the following inequalities is equivalent to Proposition 1:

$$e^{2x}-1 \geqslant (1+c^2x^2/\pi^2)\, 2xe^x,$$

$$\sum_{n=0}^{\infty} 2^n x^n/(n+1)! \geqslant (1+c^2x^2/\pi^2) \left(\sum_{n=0}^{\infty} x^n/n! \right),$$

$$\sum_{n=2}^{\infty} 2^n x^n/(n+1)! \geqslant \sum_{n=2}^{\infty} \left[\frac{1}{n!} + \frac{c^2}{\pi^2} \frac{1}{(n-2)!} \right] x^n.$$

An induction argument on n shows, however, that

$$2^n/(n+1)! \geqslant \frac{1}{n!} + \frac{c^2}{\pi^2} \frac{1}{(n-2)!} \qquad \text{for} \quad n \geqslant 2 \quad \text{and} \quad c \in [1, 5/4].$$

Therefore Proposition 1 is true.

PROPOSITION 2.

$$\frac{2\pi^2}{\pi^2 + (4x/3)^2} \geqslant \frac{\pi^2}{\pi^2 + (5x/4)^2} + \frac{\pi^2}{\pi^2 + (5x/3)^2}.$$

Proof.

$$\begin{aligned} 2(\pi^2 + 25x^2/9)(\pi^2 + 25x^2/16) &= 2\pi^4 + 1250\pi^2x^2/144 + 625x^2/72 \\ &\geqslant 2\pi^4 + 1137\pi^2x^2/144 + 625x^2/81 \\ &= (\pi^2 + 16x^2/9)[(\pi^2 + 25x^2/9) + (\pi^2 + 25x^2/16)]. \end{aligned}$$

Multiplying the above inequality by $\pi^2/[(\pi^2 + \frac{16}{9}x^2)(\pi^2 + \frac{25}{16}x^2)(\pi^2 + \frac{25}{9}x^2)]$, we obtain Proposition 2.

PROPOSITION 3.

$$f_{k+1,0}(t) \geqslant f_{k,0}(t) \qquad \text{for} \quad t \geqslant 0 \quad \text{and} \quad k \geqslant 3.$$

Proof. We shall argue by induction on k. For $k = 3$, we have

$$f_{4,0}(t) - f_{3,0}(t) = \frac{2\pi^2}{\pi^2 + (4t)^2} - \frac{2 \cdot 3te^{3t}}{e^{2\cdot 3t} - 1} - \frac{2 \cdot 4te^{4t}}{e^{2\cdot 4t} - 1}.$$

Set $x := 3t$. Then Propositions 1 and 2 yield that

$$\begin{aligned} f_{4,0}(t) - f_{3,0}(t) \geqslant &\left(\frac{\pi^2}{\pi^2 + (5x/4)^2} - \frac{2xe^x}{e^{2x} - 1}\right) \\ &+ \left(\frac{\pi^2}{\pi^2 + (5x/3)^2} - \frac{2(4xe^{4x/3}/3)}{e^{2\cdot 4x/3} - 1}\right) \geqslant 0. \end{aligned}$$

Suppose now that $k \geqslant 4$. Then $(k+1)/k \leqslant \frac{5}{4}$. We have

$$\begin{aligned} f_{k+1,0}(t) - f_{k,0}(t) &= \frac{2\pi^2}{\pi^2 t[(k+1)\,t]^2} - \frac{2kte^{kt}}{e^{2kt} - 1} - \frac{2(k+1)\,te^{(k+1)t}}{e^{2(k+1)t} - 1} \\ &= \left(\frac{\pi^2}{\pi^2 + [(k+1)\,t]^2} - \frac{2kte^{kt}}{2^{2kt} - 1}\right) \\ &\quad + \left(\frac{\pi^2}{\pi^2 + [(k+1)\,t]^2} - \frac{2(k+1)\,te^{(k+1)t}}{e^{2(k+1)t} - 1}\right) \geqslant 0, \end{aligned}$$

according to Proposition 1. Thus Proposition 3 is proved.

Consequently, $f_{k,j}(t) \geqslant f_{3,0}(t)$ for all $k \geqslant 3$ and $j \geqslant 0$. The remaining task of this section is to elaborate the nonnegativity of $f_{3,0}(t)$. For this we need some estimates.

PROPOSITION 4. *Let* $h(x) = ((xe^x/(e^x+1)) - \frac{1}{2}x)/x^2$. *Then* $h'(x) \leqslant 0$ *for* $x \geqslant 0$.

Proof. $h(x) = (e^x - 1)/2x(e^x + 1)$ and

$$h'(x) = \frac{1}{2}\frac{x(e^x+1)\,e^x - (e^x-1)[(e^x+1)+xe^x]}{[x(e^x+1)]^2} = \frac{1 + 2xe^x - e^{2x}}{2x^2(e^x+1)^2},$$

while

$$\begin{aligned} 1 + 2xe^x - e^{2x} &= 1 + 2x\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{2^n x^n}{n!} \\ &= -\sum_{n=3}^{\infty}\frac{2(2^{n-1}-n)}{n!}x^n \leqslant 0 \qquad \text{for} \quad x \geqslant 0. \end{aligned}$$

PROPOSITION 5.

$$-xe^x/(e^x-1) \geqslant -1 - \tfrac{1}{2}x - \tfrac{1}{12}x^2.$$

Proof.

$$\begin{aligned} &\left(1 + \frac{1}{2}x + \frac{1}{12}x^2\right)(e^x - 1) - xe^x \\ &= \sum_{n=2}^{\infty}\frac{x^{n+1}}{n!} - \sum_{n=1}^{\infty}\frac{1}{2}\frac{x^{n+1}}{n!} + \sum_{n=1}^{\infty}\frac{1}{12}\frac{x^{n+2}}{n!} \\ &= \sum_{n=5}^{\infty}\frac{1}{n!}\frac{n^2 - 7n + 12}{12}x^n \geqslant 0 \qquad \text{for} \quad x \geqslant 0. \end{aligned}$$

Now we are in a position to prove that $f_{3,0}(t) \geqslant 0$ for $t \in [0, 0.3]$. Write

$$\begin{aligned} f_{3,0}(t) = 2&\left(1 + \frac{te^t}{e^t+1} - \frac{te^t}{e^t-1} - \frac{t^2}{t^2+\pi^2}\right) \\ &+ 2\left(\frac{2te^{2t}}{e^{2t}+1} - \frac{2te^{2t}}{e^{2t}-1} - \frac{(2t)^2}{(2t)^2+\pi^2}\right) \\ &+ \left(1 + \frac{3te^{3t}}{e^{3t}+1} - \frac{3te^{3t}}{e^{3t}-1} - \frac{(3t)^2}{(3t)^2+\pi^2}\right) - \frac{9t^2}{\pi^2+9t^2}. \end{aligned}$$

It follows from Proposition 4 that, for $t \in [0, 0.3]$,

$$te^t/(e^t+1) - t/2 \geqslant h(0.3)\,t^2 \geqslant 0.248t^2,$$
$$2te^{2t}/(e^{2t}+1) - 2t/2 \geqslant h(0.6)(2t)^2 \geqslant 0.242(2t)^2,$$
$$3te^{3t}/(e^{3t}+1) - 3t/2 \geqslant h(0.9)(3t)^2 \geqslant 0.234(3t)^2.$$

In connection with Proposition 5, we obtain

$$2\left(1+\frac{te^t}{e^t+1}-\frac{te^t}{e^t-1}-\frac{t^2}{t^2+\pi^2}\right) \geqslant 2\left(0.248-\frac{1}{12}-\frac{1}{\pi^2}\right)t^2$$
$$\geqslant 0.126t^2,$$

$$2\left(1+\frac{2te^{2t}}{e^{2t}+1}-\frac{2te^{2t}}{e^{2t}-1}-\frac{(2t)^2}{(2t)^2+\pi^2}\right) \geqslant 2\left(0.242-\frac{1}{12}-\frac{1}{\pi^2}\right)(2t)^2$$
$$\geqslant 0.458t^2,$$

$$1+\frac{3te^{3t}}{e^{3t}+1}-\frac{3te^{3t}}{e^{3t}-1}-\frac{(3t)^2}{(3t)^2+\pi^2} \geqslant \left(0.234-\frac{1}{12}-\frac{1}{\pi^2}\right)(3t)^2$$
$$\geqslant 0.444t^2,$$

$$-\frac{9t^2}{\pi^2+9t^2} \geqslant -\frac{9t^2}{\pi^2} \geqslant -0.912t^2.$$

As a conclusion, $f_{3,0}(t) \geqslant (0.126+0.458+0.444-0.912)\,t^2 = 0.116t^2$. This shows that

$$f_k'(t) \geqslant 0 \qquad \text{for} \quad t \in [0, 0.3].$$

The next case we are going to treat is that of $t \in [0.3, 3]$. Let

$$v(t) := \frac{2\pi^2}{t^2+\pi^2} + \frac{2\pi^2}{4t^2+\pi^2} + \frac{\pi^2-9t^2}{\pi^2+9t^2},$$
$$w(t) := \frac{4te^t}{e^{2t}-1} + \frac{8te^{2t}}{e^{4t}-1} + \frac{6te^{3t}}{e^{6t}-1}.$$

Then $f_{3,0}(t) = v(t) - w(t)$. It is easily seen that $v'(t) \leqslant 0$ for $t \in [0, \infty)$. We claim that $w'(t) \leqslant 0$ $(0 \leqslant t < \infty)$, too. This is guaranteed by

PROPOSITION 6. *Let* $g(x) := 2xe^x/(e^{2x}-1)$. *Then* $g'(x) \leqslant 0$ *for* $x \geqslant 0$.

Proof. $g'(x) = -(2e^x/(e^{2x}-1)^2)(1+x+xe^{2x}-e^{2x})$, while

$$1+x+xe^{2x}-e^{2x} = (1+x)-(1-x)\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$$
$$= \sum_{n=3}^{\infty}\frac{(n-2)\,2^{n-1}}{n!}x^n \geqslant 0 \qquad \text{for} \quad x \geqslant 0.$$

Accordingly,

$$f_{3,0}(t) = v(t) - w(t) \geqslant v(b) - w(a) \qquad \text{for} \quad t \in [a, b] \ \text{ with } \ 0 < a < b. \tag{20}$$

To determine the positivity of $f_{3,0}$ we wrote a Fortran program and found that

$$v\left(\frac{n+1}{100}\right) - w\left(\frac{n}{100}\right) \geqslant 0.001 \qquad \text{for} \quad n = 30, 31, ..., 299.$$

Thus by (20) we assert that

$$f_{3,0}(t) > 0 \qquad \text{for} \quad t \in \left[\frac{n}{100}, \frac{n+1}{100}\right], \qquad n = 30, 31, ..., 299.$$

Therefore

$$f_{3,0}(t) > 0 \qquad \text{for} \quad t \in \bigcup_{n=30}^{299}\left[\frac{n}{100}, \frac{n+1}{100}\right] = [0.3, 3].$$

So far we have shown that $\Omega_k'(q) \geqslant 0$ for $q \in [1, 20]$.

4. The Monotonicity of $\Omega_k(q)$ for $q \in [20, \infty)$

Let

$$f(q) := (-1)^k\,(2k-1)!\,q^k[0, 1, ..., 2k-1]\frac{1}{q^{\cdot} + q^k}. \tag{21}$$

Then

$$f(q) = (-1)^k\,q^k\sum_{l=0}^{2k-1}\binom{2k-1}{l}(-1)^{l+1}\frac{1}{q^l + q^k}$$
$$= (-1)^k\sum_{l=0}^{2k-1}(-1)^{l+1}\binom{2k-1}{l}\frac{1}{1+q^{l-k}}$$

$$= \sum_{l=0}^{k-1} (-1)^{k+l+1} \binom{2k-1}{l} \frac{q^{k-l}}{1+q^{k-l}}$$

$$+ (-1) \binom{2k-1}{k} \frac{1}{2} + \sum_{l=k+1}^{2k-1} (-1)^{k+l+1} \binom{2k-1}{l} \frac{1}{1+q^{l-k}}.$$

It follows that

$$f'(q) = \sum_{l=0}^{k-1} (-1)^{k+l+1} \binom{2k-1}{l} \frac{(k-l)\, q^{k-l-1}}{(1+q^{k-l})^2}$$

$$+ \sum_{l=k+1}^{2k-1} (-1)^{k+l} \binom{2k-1}{l} \frac{(l-k)\, q^{l-k-1}}{(1+q^{l-k})^2}$$

$$= \sum_{l=1}^{k} (-1)^{l-1} \binom{2k-1}{k-l} \cdot \frac{lq^{l-1}}{(1+q^l)^2}$$

$$- \sum_{l=1}^{k-1} (-1)^{l-1} \binom{2k-1}{k+l} \cdot \frac{lq^{l-1}}{(1+q^l)^2}$$

$$= \sum_{l=1}^{k-1} (-1)^{l-1} \left[\binom{2k-1}{k-l} - \binom{2k-1}{k+l} \right] \frac{lq^{l-1}}{(1+q^l)^2}$$

$$+ (-1)^{k-1} \frac{kq^{k-1}}{(1+q^k)^2}. \tag{22}$$

Now we need the following propositions.

PROPOSITION 7.

$$\left[\binom{2k-1}{k-l} - \binom{2k-1}{k+l} \right] \frac{lq^{l-1}}{(1+q^l)^2}$$

decreases as l increases and $q \geqslant 6$.

Proof.

$$\binom{2k-1}{k-l} - \binom{2k-1}{k+l} = \frac{(2k-1)!}{(k-l)!\,(k-1+l)!} \left(1 - \frac{k-l}{k+l}\right)$$

$$= \frac{(2k-1)!}{(k-l)!\,(k+l)!}\, 2l. \tag{23}$$

We want to show

$$\frac{(2k-1)!}{(k-l)!\,(k+l)!}\, 2l \frac{lq^{l-1}}{(1+q^l)^2}$$

$$\geqslant \frac{(2k-1)!}{(k-l-1)!\,(k+l+1)!}\, 2(l+1) \frac{(l+1)\, q^l}{(1+q^{l+1})^2} \qquad \text{for} \quad q \geqslant 6. \tag{24}$$

It is easily seen that (24) is equivalent to

$$\frac{1}{q}\left(\frac{q^{l+1}+1}{q^l+1}\right)^2 \geqslant \frac{k-l}{k+l+1}\left(1+\frac{1}{l}\right)^2;$$

however,

$$\frac{1}{q}\left(\frac{q^{l+1}+1}{q^l+1}\right)^2 = \frac{q^{2l+2}+2q^{l+1}+1}{q(q^{2l}+2q^l+1)} \geqslant q-2q^{l-1} \geqslant q-2,$$

because

$$\begin{aligned}(q-2q^{1-l})\,q(q^{2l}+2q^l+1) &= q^{2l+2}-q^2-2q^{2-l}\\ &\leqslant q^{2l+2}+2q^{l+1}+1.\end{aligned}$$

Meanwhile

$$\frac{k-l}{k+l+1}\left(1+\frac{1}{l}\right)^2 \leqslant 4.$$

Therefore (24) holds for $q \geqslant 6$, and Proposition 7 is proved.

PROPOSITION 8. *For $k \geqslant 2$ and $q \geqslant 6$,*

$$f'(q) \geqslant \binom{2k-1}{k-1}\frac{2}{k+1}\frac{1}{(1+q)^2} - \binom{2k-1}{k-2}\frac{4}{k+2}\frac{2q}{(1+q^2)^2}. \tag{25}$$

In particular, $f'(q) \geqslant 0$ and

$$f(q) \leqslant \lim_{q\to\infty} f(q) = \binom{2k-1}{k-1}\frac{1}{2(2k-1)}. \tag{26}$$

Proof. Suppose first k is even, $k=2m$. Then (22) and (23) yield that

$$\begin{aligned} f'(q) = {} & \binom{2k-1}{k-1}\frac{2}{k+1}\frac{1}{(1+q)^2} - \binom{2k+1}{k-2}\frac{4}{k+2}\frac{2q}{(1+q^2)^2}\\ & + \sum_{j=2}^{m-1}\left[\frac{(2k-1)!}{(k-2j+1)!\,(k+2j-2)!}\frac{2(2j-1)}{k+2j-1}\frac{(2j-1)\,q^{2j-2}}{(1+q^{2j-1})^2}\right.\\ & \left. - \frac{(2k-1)!}{(k-2j)!\,(k+2j-1)!}\frac{2\cdot 2j}{k+2j}\frac{2jq^{2j-1}}{(1+q^{2j})^2}\right]\\ & + \left[(2k-2)\frac{(k-1)\,q^{k-2}}{(1+q^{k-1})^2} - \frac{kq^{k-1}}{(1+q^k)^2}\right]. \end{aligned}$$

By Proposition 7 all the terms under the summation sign are positive. Moreover,

$$q^{k-2}/(1+q^{k-1})^2 \geqslant q^{k-1}/(1+q^k)^2 \qquad \text{for} \quad q \geqslant 1,$$

and

$$(2k-2)(k-1)\frac{q^{k-2}}{(1+q^{k-1})^2} - \frac{kq^{k-1}}{(1+q^k)^2}$$
$$\geqslant [(2k-2)(k-1)-k]\frac{q^{k-1}}{(1+q^k)^2} \geqslant 0 \qquad \text{for} \quad k \geqslant 2.$$

For odd k, the proof is similar. Thus (25) holds. Furthermore,

$$f'(q) \geqslant \frac{2}{(1+q)^2}\frac{(2k-1)!}{(k-2)!\,(k+2)!}\left[\frac{k+2}{k-1} - \frac{4(1+1/q)^2}{q(1+1/q^2)^2}\right]$$
$$\geqslant \frac{2}{(1+q)^2}\frac{(2k-1)!}{(k-2)!\,(k+2)!}\left[1-4\left(1+\frac{1}{6}\right)^2\right]$$
$$\geqslant 0 \qquad \text{fcr} \quad q \geqslant 6,$$

and

$$f(q) \leqslant \lim_{q\to\infty} f(q) = \lim_{q\to\infty}\left[\binom{2k-1}{k-1}\frac{1}{2}\Omega_k(q)\right]$$
$$= \binom{2k-1}{k-1}\frac{1}{2(2k-1)}.$$

PROPOSITION 9. *Let*

$$S(q) = 4\sum_{v=1}^{k-1}\frac{vq^{v-1}}{q^{2v}-1} + 2\frac{kq^{k-1}}{q^{2k}-1}.$$

Then

$$S(q) \leqslant \frac{4q^2}{(q^2-1)(q-1)^2} \qquad \textit{for} \quad q \geqslant 1.$$

Proof. We have

$$S(q) = 4\sum_{v=1}^{k-1}\frac{q^{2v}}{q^{2v-1}}vq^{-(v+1)} + 2kq^{-(k+1)}\frac{q^{2k}}{q^{2k}-1}.$$

Note that

$$\frac{q^{2v}}{q^{2v}-1} \leqslant \frac{q^2}{q^2-1} \qquad \text{for} \quad v \geqslant 1 \quad \text{and} \quad q \geqslant 1.$$

Hence

$$S(q) \leqslant \frac{4q^2}{q^2-1} \sum_{v=1}^{\infty} vq^{-(v+1)} = \frac{4q^2}{q^2-1} \frac{d}{dq}\left(-\sum_{v=1}^{\infty} q^{-v}\right)$$
$$= \frac{4q^2}{q^2-1} \frac{d}{dq}\left(\frac{-1}{1-q^{-1}}\right) = \frac{4q^2}{(q^2-1)(q-1)^2}.$$

PROPOSITION 10. *Let*

$$g_k(q) = \frac{2k-1}{k+1}\left[1 - \frac{4(k-1)}{k+2} \frac{q}{1+q^2}\right].$$

Then

$$g_{k+1}(q) \geqslant g_k(q) \qquad k = 1, 2, 3, \ldots, \quad q \geqslant 12.$$

Proof.

$$g_{k+1}(q) - g_k(q) = \frac{3}{(k+1)(k+2)(k+3)}\left[(k+3) - (3k-1)\, 4 \frac{q}{1+q^2}\right]$$
$$\geqslant \frac{3}{(k+1)(k+1)(k+3)}\left[(k+3) - (3k-1)\, 4 \frac{12}{1+12^2}\right]$$
$$\geqslant 0 \qquad \text{for} \quad q \geqslant 12.$$

Now we are in a position to prove the monotonicity of $\Omega_k(q)$ for $q \in [20, \infty)$. From (16) and (21) we see that

$$\Omega_k(q) = 2k!\,(k-1)!/(2k-1)! \prod_{m=1}^{k} \frac{q^m+1}{q^m-1} \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} f(q).$$

Hence

$$\frac{\Omega_k'(q)}{\Omega_k(q)} = -\left(4 \sum_{v=1}^{k-1} \frac{vq^{v-1}}{q^{2v}-1} + 2\frac{kq^{k-1}}{q^{2k}-1}\right) + \frac{f'(q)}{f(q)} = -S(q) + \frac{f'(q)}{f(q)}.$$

By Proposition 9 we have

$$S(q) \leqslant \frac{1}{q^2} \frac{4q^4}{(q^2-1)(q-1)^2} \leqslant \frac{1}{q^2} \frac{4 \cdot 20^4}{(20^2-1)(20-1)^2} \leqslant \frac{4.45}{q^2} \quad \text{for } q \geqslant 20.$$

Moreover, Propositions 8 and 10 tell us that

$$\begin{aligned}
\frac{f'(q)}{f(q)} &\geqslant \frac{1}{(1+q)^2}\frac{4(2k-1)}{k+1}\left[1-\frac{4(k-1)}{k+2}\frac{q}{1+q^2}\right]=\frac{4}{(1+q)^2}\,g_k(q)\\
&\geqslant \frac{4}{(1+q)^2}\,g_4(q)=\frac{4}{(1+q)^2}\frac{7}{5}\left(1-\frac{2q}{1+q^2}\right)\\
&=\frac{1}{q^2}\frac{28}{5}\frac{(1-1/q)^2}{(1+1/q)^2}\frac{1}{1+1/q^2}\\
&\geqslant \frac{1}{q^2}\frac{28}{5}\frac{(1-(1/20))^2}{(1+(1/20))^2}\frac{1}{1+1/20^2}\\
&\geqslant \frac{4.55}{q^2} \qquad \text{for} \quad q\geqslant 20 \quad \text{and} \quad k\geqslant 4.
\end{aligned}$$

Therefore

$$\frac{\Omega_k'(q)}{\Omega_k(q)}=\frac{f'(q)}{f(q)}-S(q)\geqslant\frac{4.55}{q^2}-\frac{4.45}{q^2}=\frac{0.1}{q^2}>0$$
$$\text{for} \quad k\geqslant 4 \quad \text{and} \quad q\geqslant 20.$$

It remains to check the case $k=3$. For this we shall make a straightforward computation:

$$\begin{aligned}
\Omega_3(q) &= 24\left(\frac{q+1}{q-1}\right)^2\left(\frac{q^2+1}{q^2-1}\right)^2\frac{q^3+1}{q^3-1}\,q^3(-1)[0,1,2,3,4,5]\frac{1}{q^{\cdot}+q^3}\\
&=\frac{24}{120}\left(\frac{q+1}{q-1}\right)^2\left(\frac{q^2+1}{q^2-1}\right)^2\frac{q^3+1}{q^3-1}\,q^3\left(\frac{1}{1+q^3}-5\frac{1}{q+q^3}\right.\\
&\quad\left.+10\frac{1}{q^2+q^3}-10\frac{1}{2q^3}+5\frac{1}{q^3+q^4}-\frac{1}{q^3+q^5}\right)\\
&=\frac{1}{5}\frac{q^2+1}{q^2+q+1}.
\end{aligned}$$

Thus

$$\Omega_3'(q)=\frac{1}{5}\frac{q^2-1}{(q^2+q+1)^2}\geqslant 0 \qquad \text{for} \quad q\geqslant 1.$$

This completes the proof of Theorem 1.

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